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LETTER TO THE EDITOR

Natural spectral line widths in undamped many-wave systems

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Abstract. Edwards' approach to turbulence is re-interpreted to allow the determination of the natural spectral line width of a coupled many-wave system in non-thermal equilibrium. The method is applied to the non-resonant feedback laser for which an analytic expression for the line width is given. It is also shown that above the threshold region the wave–wave coupling provides the dominant noise source.

An undamped wave (self-sustained oscillation) produced by a non-linear instability mechanism has zero spectral line width in the context of a deterministic description. In reality, the line width is, of course, finite. Even if one eliminates all external perturbations a lower bound to the line width is given by the natural (or intrinsic) line width.

From the theory of the single-mode laser it is known that the natural line width comes about by internal noise such as spontaneous emission processes which necessitate a stochastic description. Such a formulation can be given using generalized Langevin equations (Haken 1966, Lax 1967), the Fokker–Planck (FP) approach (Risken 1966, Lax 1967), or density matrix equations (Scully and Lamb 1967, Gordon 1967). These methods successfully established all statistical properties of a single undamped wave as realized in the single-mode laser (for reviews cf Arecchi and Schulz-Dubois 1972, Risken 1970, Haken 1970). Specifically the spectral line width was found to be given by

$$\Delta\omega = \alpha_{\mathrm{L}}(a)\frac{q}{\langle I\rangle}, \tag{1}$$

where $\langle I\rangle$ is the mean intensity, q the total strength of the fluctuating forces appearing in the Langevin equation for the amplitude and $\alpha_{\mathrm{L}}(a)$ the line width factor, which as a function of the pumping parameter a (for its definition see Risken 1966) decreases monotonically from the value 2 far below threshold ($a \ll -1$) to the value 1 far above threshold ($a \gg 1$). The problem then arises how to extend the result of equation (1) to the case of several undamped waves, which since they occur at a state far from thermal equilibrium are strongly coupled.

Nobody has managed so far to apply the above mentioned basic methods to the spectral line width problem of coupled multi-wave systems. Especially, it seems rather hopeless to solve the multi-wave non-stationary FP equation since even the single-mode equation requires computer calculations (Hempstead and Lax 1967, Risken and Vollmer 1967). It is therefore necessary to adopt approximation schemes similar to those of many-body theory.

Richter and Grossmann (1972) have put forward a linear response formalism in analogy to the dissipation fluctuation theorems of equilibrium statistical mechanics but with the 'noise energy' q replacing $k_B T$. Their method gives reasonable results for few-wave systems but it cannot be extended to the many-wave case. The many-wave case, however, should permit simplifications due to statistical limit theorems. Brunner and Paul (1969) apply heuristic averaging procedures to the Heisenberg equations of motion of the coupled multi-wave system of a non-resonant feedback laser (Ambartsumyan *et al* 1969) and obtain, in the threshold region, a form for $\Delta\omega$ as given by equation (1) with $\langle I \rangle$ = intensity of a *single* mode and

$$\alpha_L \equiv 2. \tag{2}$$

This is the usual result for a linearly damped gaussian process. Since the individual modes of a non-resonant feedback laser are thermal (Ambartsumyan *et al* 1967), the result (equation (2)) seems quite natural. However, it should be pointed out that q is still the original fluctuation strength of each mode. Sufficiently far above threshold one does, however, expect a renormalization of q to occur as a consequence of the wave–wave coupling.

In this letter, we sketch a method which is capable of solving this problem for the case of $N \gg 1$ coupled waves. The method is a modification of an approach used in turbulence theory by Edwards (1964) and Edwards and McComb (1969). Edwards attacks one of the central problems of turbulence theory, namely the determination of the wavevector dependence of $\langle u_k u_{-k} \rangle$, ie of the Fourier transform of the (spatial) velocity correlation function. He starts by setting up the FP equation for the large number of coupled waves u_k. He then derives equations for $\langle u_k u_{-k} \rangle$ involving the stationary solution $P(\{u_k\})$ of this FP equation. The physical basis of his approach is provided by the fact that through the wave–wave coupling every wave becomes thermal, so that a gaussian distribution P_0 is the appropriate zeroth-order approximation to P. P_0 is associated with a renormalized free wave FP operator with effective damping constants $\tilde{D}_k$ and renormalized fluctuation strengths $\tilde{q}_k$. These are related to the wave intensity $\langle u_k u_{-k} \rangle$ by a relation analogous to equation (1)

$$\tilde{D}_k = \frac{2\tilde{q}_k}{\langle u_k u_{-k} \rangle}. \tag{1'}$$

One of the missing two equations between these three quantities is provided by the energy balance equation which formally follows from the requirement that $\Delta P = P - P_0$ does not contribute to $\langle u_k u_{-k} \rangle$. A second equation has been proposed by Edwards and McComb (1969) using the principle of maximum entropy. One then ends up with a set of two coupled non-linear integral equations which still have to be solved.

Returning to our problem of the spectral line width of well behaved many-wave systems we note a few points in which our problem differs from and, in fact, becomes simpler than the turbulence problem. The wave–wave coupling in turbulence theory is provided by the $\boldsymbol{u}$ grad $\boldsymbol{u}$ non-linearity of the Navier–Stokes equations. In Fourier space this gives a parametric type of wave–wave coupling and causes a cascade energy transfer from low wavevectors to high wavevectors. Such a coupling is inappropriate for many-wave systems like that of the non-resonant feedback laser. There the wave frequencies are almost identical and the interaction comes about by absorption plus subsequent re-emission of photons. This gives rise to a tri-linear interaction such that the drift vector in the FP equation is given by

$$D_\nu = \left(D_\nu^{(0)} - \sum_{\nu', \nu''} M_{\nu\nu'\nu''\nu} u_{\nu'} u_{\nu''} \right) u_\nu .$$

The index ν now numbers the modes, $D_\nu^{(0)}$ is the linear gain and M the coupling matrix. A tri-linear interaction ensures self-stabilization of the waves and makes it easier to compute $\langle u_\nu u_{-\nu}\rangle$ by other means. In fact, we may assume that the 'spectrum' $\langle u_\nu u_{-\nu}\rangle$ is known. In this case the equivalent of the energy balance equation for tri-linear interactions together with equation (1′) provide us with a closed set of equations for $\tilde{D}_\nu$ and $\tilde{q}_\nu$.

The relation to the spectral line width problem finally is provided by the wave equivalent to the quasi-particle hypothesis which states that in a well behaved (non-turbulent) many-wave system, the low-lying excitations are weakly damped and associated with just one decay constant (exponential decay framework). In our case this permits the identification

$$\tilde{D}_\nu = \Delta\omega_\nu. \tag{3}$$

The equivalent to the energy balance equation then becomes a closed integral equation for the spectral line widths of the many-wave system. We do not go into the details of the derivation which will be given elsewhere.

The equation obtained in this way admits a solution for the non-resonant feedback laser where the 'spectrum' is practically mode number independent and given by (in *scaled* notation cf Rowlands and Wonneberger, to be published)

$$\langle z\rangle = \sqrt{2}\frac{D_{-(N+1)}(-Na/\sqrt{2})}{D_{-N}(-Na/\sqrt{2})}. \tag{4}$$

D denotes parabolic cylinder functions. Using this formula, the revised value of the line width factor turns out to be given by

$$\alpha_L(a, N) = 2 + a\langle z\rangle. \tag{5}$$

For $a \ll 1$ one has $\langle z\rangle \ll 1$ for $N \gg 1$. Thus the second term may be neglected and the result as given by equation (2) is re-obtained. For the practically important case $a \gg 1$ we obtain

$$\alpha_L = a^2. \tag{6}$$

This demonstrates that the wave–wave coupling has completely changed the fluctuation strength to much larger values giving a much larger natural line width.

The intensity fluctuation line width $\Delta\omega_I$ of each individual wave equals $2\Delta\omega$ since the wave obeys gaussian statistics. It is noted that $\Delta\omega_I$ behaves similarly to the effective intensity fluctuation line width of a single-mode laser (Jakeman and Pike 1971).

Finally, a qualitatively different result predicted by equation (5) as compared to equation (2) is pointed out. For $q \to 0$, ie the vanishing of the intrinsic noise (thermal noise, vacuum fluctuations and spontaneous emission noise), the spectral line width of the single-mode laser (as well as that of the non-resonant feedback laser in the Brunner–Paul approximation) vanishes. This is not true for $\Delta\omega$ using α_L according to equation (5). For $q \to 0$, $a \to \infty$, $\alpha_L \to a^2$, but a^2q remains finite. The physical reason for this is that now the wave–wave coupling provides the reservoir and the associated noise for every wave to have non-vanishing spectral line width. Under many experimental situations one has $a > 1$ and it is this mechanism which then accounts for the natural spectral line width of a coupled many-wave system.

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References

Ambartsumyan R V, Kryukov P G, Letokhov V S and Matveets Yu A 1967 *Zh. Eksp. Teor. Fiz.* **53** 1955–66 (1967 *Sov. Phys.-JETP* **26** 1109–14)
Ambartsumyan R V, Basov N G, Kryukov P G and Letokhov V S 1969 *Progress in Quantum Electronics* vol 1/3, eds J H Sanders and K W H Stevens (Oxford: Pergamon) pp 107–85
Arecchi F T and Schulz-Dubois E O (eds) 1972 *Laser Handbook* vol 1 (Amsterdam: North-Holland)
Brunner W and Paul H 1969 *Ann. Phys., Lpz* **23** 384–96
Edwards S F 1964 *J. Fluid Mech.* **18** 239–73
Edwards S F and McComb W D 1969 *J. Phys.* A: *Gen. Phys.* **2** 157–71
Gordon J P 1967 *Phys. Rev.* **161** 367–86
Haken H 1966 *Z. Phys.* **190** 327–56
—— 1970 *Encyclopedia of Physics* vol 25/2c, ed S Flügge (Berlin Springer)
Hempstead R D and Lax M 1967 *Phys. Rev.* **161** 350–66
Jakeman E and Pike E R 1971 *J. Phys.* A: *Gen. Phys.* **4** L56–7
Lax M 1967 *Phys. Rev.* **157** 213–31
Richter P H and Grossmann S 1972 *Z. Phys.* **255** 59–75
Risken H 1966 *Z. Phys.* **191** 302–12
—— 1970 *Progress in Optics* vol 8, ed E Wolf (Amsterdam: North-Holland) pp 239–94
Risken H and Vollmer H D 1967 *Z. Phys.* **201** 323–30
Scully M O and Lamb W E Jr 1967 *Phys. Rev.* **159** 208–26